Universitat de Barcelona - Department of Economic Theory
Master in Economics

Macroeconomics I - Fall Semester 2011

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Problem Set 5.

Stochastic Growth with a time-homogeneous Markov chain process for technology.

Consider the Stochastic Growth model we saw in class. Production is

$$Y_t = z_t F(K_t, L_t),$$

where F is Neoclassical and the technology z_t is the realization in t of the process $\tilde{z}_t$. Suppose that $\tilde{z}_t$ is a time-homogeneous Markov chain with support set $Z = \{z^l, z^h\}$, where $z^l > 0$, $z^h > 0$, and $z^h > z^l$. The probability of $\tilde{z}_t = z^l$ is $1/3$ if $\tilde{z}_{t-1} = z^l$ and $1/2$ if $\tilde{z}_{t-1} = z^h$, for every t.

The utility function is

$$\mathscr{U}_0 = \sum_{t=0}^{T} \beta^t \sum_{z^t \in Z^{t+1}} Pr(z^t) U(c_t(z^t)), \tag{1}$$

where $Pr(z^t)$ is the probability that the history in period t is z^t. Suppose that the realization of $\tilde{z}_t$ in period $t = 0$ is given and equal to z^l, and that the initial capital is given and $k_0 > 0$.

1. What are the possible histories of the technology in $t = 0$, $t = 1$, and $t = 2$, that is, z^0, z^1, and z^2? Derive explicitly the probability of each possible history up to time $t = 2$. Isolate from the utility function in (1) the per-period utility in period 0 and the expected per-period

utilities in periods $t = 1$ and $t = 2$. Express these per-period utilities using the probabilities of each possible history. [Remember that the expected utility is the sum of the utility coming from the consumption at each history weighted by the probability of that history.]

2. How many resource constraints exist in $t = 0$ and $t = 1$? Write down the problem of the Central Planner listing explicitly the constraints in $t = 0$ and $t = 1$, and leaving the indexation to t for $t \geq 2$.[Remember that the first realization of $\tilde{z}_t$ is given.]

3. Take the FOC with respect to the control variables in $t = 0$ and $t = 1$ for all possible histories. How many FOC's are there for $t = 0, 1$?

4. Derive the equation that relates consumption in $t = 0$ with consumption and capital in $t = 1$ for any possible history in $t = 1$. List also the binding budget constraints for $t = 0, 1$.

5. Generalize the previous point in order to obtain the Euler equation and the binding budget constraint for every t.[Express explicitly the conditional expectation using the properties of the time-homogeneous Markov chain assumed above.]

6. Suppose that in the decentralized economy we have a representative household. The budget constraint of the household is

$$c_t(z^t) + k_{t+1}(z^t) \leq w_t(z^t) + (r_t(z^t) + 1 - \delta)k_t(z^{t-1}), \tag{2}$$

for every t, for every $z^t \in Z^{t+1}$. Note that we neglect the exchange of bonds across households. The profits of the representative firm are

$$\Pi_t = z_t F(K_t, L_t) - r_t K_t - w_t L_t, \tag{3}$$

for every t. The firm maximizes profits taking as given the prices and the level of technology in t, z_t. Define an equilibrium in this economy.

7. Derive the FOCs of the household's problem and of the firm's problem. Is the equilibrium Pareto-optimal? [Derive both the Euler condition and the binding resource constraint from the equilibrium allocation.]