Universitat de Barcelona - Department of Economic Theory
Master in Economics

Macroeconomics I - Fall Semester 2011

Lorenzo Burlon

Problem Set 2. The Ramsey-Cass-Koopmans model.

The Ramsey-Cass-Koopmans model with population growth and Harrod-neutral technological progress.

Consider the Neoclassical growth model we saw in class. On the preference side, we allow for a representative household. The preferences of each member of the household are defined over the sequence of consumption $\{\tilde{c}_t\}_{t=0}^{\infty}$, with $\tilde{c}_t \equiv C_t/N_t$ where C_t is the consumption of the whole household and N_t is the household size (how many members the household has). The preferences of the single household member are represented by the utility function $\mathscr{U} : \mathbb{R}^{\infty} \to \mathbb{R}$. Denote $\mathscr{U}_t$ the utility that corresponds to the sequence of consumption that starts in period t. Suppose that preferences are recursive and additively separable, so that there exists a per-period utility function $U : \mathbb{R} \to \mathbb{R}$ and a scalar ρ such that $\mathscr{U}_t = U(\tilde{c}_t) + \rho\mathscr{U}_{t+1}$. Assume that $\rho \in (0, 1)$ and that U is Neoclassical. Moreover, each member of the household is endowed with 1 unit of labor, so that the total labor endowment of the household is equal to the size of the household. In other words, $l_t \leq 1$ and $L_t = N_t l_t \leq N_t$. The size N_t of the household varies over time according to

$$N_{t+1} = (1 + n)N_t,$$

for every t and with the initial size N_0 as given.

On the production side, we consider a production function with labor-augmenting technology,

$$Y_t = F(K_t, A_t L_t) = K_t^{\alpha} (A_t L_t)^{1-\alpha}.$$

with $\alpha \in (0,1)$. The resource constraint of this economy is

$$C_t + I_t \leq Y_t,$$

where I_t is aggregate investment. Capital evolves according to the usual law of motion,

$$K_{t+1} = I_t + (1-\delta)K_t,$$

for every t, with K_0 taken as given. Moreover, the technology A_t grows at the exogenous rates g with the initial level A_0 of technology as given.

1. Define and characterize in aggregate terms a feasible allocation in this economy. [Pay attention not to mix labor supply with household size and remember to include the natural nonnegativity constraints.]

2. Define and characterize a feasible allocation in this economy according to the normalization
$$k_t \equiv \frac{K_t}{A_t L_t}.$$

3. The utility $\mathscr{U}_\tau$ of the single household member in period τ is
$$\mathscr{U}_\tau = U(\tilde{c}_\tau) + \rho \mathscr{U}_{\tau+1}.$$
Hence, the utility of the whole household in time τ is
$$N_\tau \mathscr{U}_\tau = N_\tau U(\tilde{c}_\tau) + \rho N_\tau \mathscr{U}_{\tau+1}.$$
What is the utility of the household in period $\tau = 0$, that is, $N_0 \mathscr{U}_0$, as a function of $\{\tilde{c}_t\}_{t=0}^{\infty}$?

4. Let us normalize $N_0 = 1$. Recall from part (b) that
$$c_t \equiv \frac{C_t}{A_t L_t},$$
and that $\tilde{c}_t \equiv C_t/N_t$. Express $\tilde{c}_t$ as a function of A_t, L_t, N_t, and c_t. Substitute $\tilde{c}_t$ in the utility of the household in the initial period, that

is, express $\mathscr{U}_0$ as a function of the sequence $\{c_t\}_{t=0}^{\infty}$. Formulate the Ramsey problem and define a Ramsey-optimal allocation given the initial capital per efficiency unit of labor k_0. Explain why we can simplify the problem in terms of the labor supply of the household by setting $L_t = N_t$ for every t. Formulate the simplified Ramsey problem.

5. Suppose a Constant-Relative-Risk-Aversion (CRRA) per-period utility function, that is,
$$U(\tilde{c}_t) = \frac{\tilde{c}_t^{1-\theta}}{1-\theta},$$
with $\theta > 0$. Show that the objective function of the Ramsey problem boils down to
$$\mathscr{U}_0 = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\theta}}{1-\theta},$$
where
$$\beta = \frac{\rho(1+g)^{1-\theta}}{(1+n)} \approx log(\rho) + (1-\theta)g - n.$$
What is the necessary condition on ρ, n, θ, and g in order to have a finite initial utility? [Normalize A_0 to 1.]

6. Derive and list the Karush-Kuhn-Tucker conditions, that is, the necessary conditions for the optimal solution of the simplified Ramsey problem. [You can skip the conditions regarding the natural nonnegativity constraints.]

7. Derive the system of difference equations that govern the dynamics of c_t and k_t and identify the conditions that assure existence and uniqueness of a solution. [Express c_{t+1} and k_{t+1} as functions of c_t and k_t using the stationarity conditions and state explicitly the initial condition and the transversality condition.]

8. Define and characterize the steady state(s) in this economy.

9. Analyze the dynamics of c_t and k_t and represent them on a phase diagram. Identify graphically the saddle path and argue why it is unique.

10. Determine the growth rate of the aggregate variables when the economy lies on its balanced growth path.